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Abstract: Through two different approaches, this report proposes two general controllers for unicycle-type and two-steering-wheels mobile robots. For both systems, conditions for asymptotical convergence to a predefined path are established and simulation results are presented. Rather than writing the systems' equations with respect to a fixed reference frame, the robot state is here parametrized relative to the followed path, in terms of distance and orientation.

Key-words: mobile robots, nonholonomic constraints, feedback linearization, Lyapunov design, Frenet frame

(Résumé : tsvp)

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Poursuite de trajectoires pour véhicules de type unicycle et robots mobiles à deux trains directeurs

Résumé : Dans ce rapport, deux approches sont utilisées pour la synthèse de lois de commande pour des véhicules de type unicycle et des robots mobiles équipés de deux trains directeurs. Des conditions de convergence asymptotique vers la trajectoire désirée sont données et des résultats de simulation sont présentés. Une originalité des modèles considérés et des lois de commande proposées provient de la paramétrisation de l'attitude des véhicules en termes de distance et d'orientation relativement à la trajectoire suivie.

Mots-clé : robots mobiles, non-holonomie, linearisation par retour d'état, méthodes de Lyapunov, repère de Frenet

1 Introduction

Several articles and reports have been written over the past ten years, on the problem of controlling wheeled robots. Following simple control laws, based on tangential linearization or heuristic methods [2], [10], [9], [5], new and more general controllers have been proposed on the basis of nonlinear control theory [3], [4], [6], [7], [8]. This article focusses on the trajectory tracking issue for robots that move on flat ground without skidding, and is concerned with a particular approach, which was first proposed in [6]. It is based on a parametrization which separates the trajectory tracking problem from the control of the translational velocity. The approach appears to be well suited to mobile robot control, as long as the objective is not to stabilize the system about a given posture, and consists in tracking a given trajectory independently of translational speed, as in the case of road following. The paper is organized as follows :

- The solution proposed in [6] for a unicycle-type robot is first recalled and complemented.
- The approach is then extended to the case of a two-steering-wheels mobile robot, which may be seen as a unicycle-type robot with one additional degree of freedom which allows the vehicle orientation to be controlled independently of the path's direction. This family of mobile robots is of particular interest in the field of industrial handling due to their enhanced mobility capabilities.

In section 2, a state space representation is developped for both mobile robots. A linearizing feedback approach is presented in section 3 for the trajectory tracking problem. In section 4, another non-linear control is proposed via a Lyapunov analysis. For each control law, convergence conditions and simulation results are presented.

2 Modelling

2.1 Kinematic equations of a moving point

In this paragraph, system equations describing the motion of a point relative to a given curve (C) are derived. Consider a moving point M and the associated Frenet frame (T) defined on the curve as indicated in Fig.1. The point P is the orthogonal projection of the point M onto the curve. Conditions for which this projection is defined without ambiguity are given in [6]. The coordinates of M are $(0, y, 0)$ in the frame $(T) = (\mathrm{P}; \vec{m}, \vec{n}, \vec{k})$ and $(X, Y, 0)$ in the fixed reference frame $(R) = (\mathrm{O}; \vec{i}, \vec{j}, \vec{k})$.

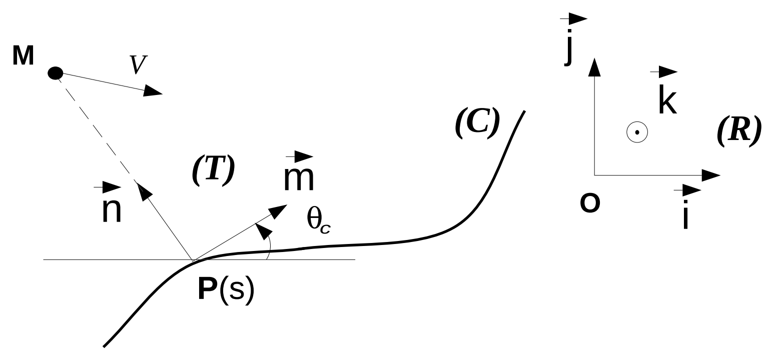


Figure 1: Frames and Notations

The signed curvilinear abscissa of (T) along the curve is denoted as s. The position of the point M in the plane $(\mathrm{O}; \vec{i}, \vec{j})$ is then characterized by the couple of cartesian coordinates (X, Y), or, equivalently by the couple of variables (s, y). In the particular case where the curve (C) coincide with the axis $(\mathrm{O}; \vec{i})$, s and y are just equal to X and Y respectively, assuming that $s = 0$ when P coincides with O.
A classical law of Mechanics gives :

$$\left(\frac{d\,\overrightarrow{\mathrm{OM}}}{dt}\right)_R = \left(\frac{d\,\overrightarrow{\mathrm{OP}}}{dt}\right)_R + \left(\frac{d\,\overrightarrow{\mathrm{PM}}}{dt}\right)_T + \vec{w}_c \wedge \overrightarrow{\mathrm{PM}} \tag{1}$$

with :

$$\left[\overrightarrow{\mathrm{PM}}\right]_T = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \quad \text{and} \quad [\vec{w}_c]_R = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_c = c_c(s)\dot{s} \end{pmatrix} \tag{2}$$

$[\vec{w_c}]_R$ is the rotation velocity vector of frame (T) with respect to (R), $\left(\frac{d}{dt}\right)_R$ means time derivation with respect to the frame (R), and $c_c(s)$ is the path's curvature at T.
Let R_T^R denote the transfer rotation matrix from (R) to (T),

$$R_T^R = \begin{pmatrix} cos\theta_c & sin\theta_c & 0 \\ -sin\theta_c & cos\theta_c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One gets from (1) :

$$\begin{pmatrix} \dot{s} \\ \dot{y} \\ 0 \end{pmatrix} = R_T^R \begin{pmatrix} \dot{X} \\ \dot{Y} \\ 0 \end{pmatrix} + \begin{pmatrix} c_c(s)y\dot{s} \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

and thus :

$$\left\{ \begin{array}{lcl} \dot{s} & = & \begin{pmatrix} cos\theta_c & sin\theta_c \end{pmatrix} . \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} / (1 - c_c(s)y) \\ \dot{y} & = & \begin{pmatrix} -sin\theta_c & cos\theta_c \end{pmatrix} . \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \end{array} \right. \tag{4}$$

2.2 Model of a unicycle-type vehicule

The kinematic equations of a unicycle-type vehicule with two actuated wheels on a common axle and the point M at mid-distance of these wheels are as follows :

$$\left\{ \begin{array}{lcl} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} & = & v. \begin{pmatrix} cos\theta_m \\ sin\theta_m \end{pmatrix} \\ \dot{\theta}_m & = & w \end{array} \right. \tag{5}$$

v and w are the mobile robot translational and angular velocities respectively, and θ_m is the vehicule's orientation with respect to the fixed frame.
Using eqns (5) in (4) gives, in terms of the (s, y) parametrization :

$$\left\{ \begin{array}{lcl} \dot{s} & = & vcos\,(\theta_m - \theta_c)\,/(1 - c_c y) \\ \dot{y} & = & vsin\,(\theta_m - \theta_c) \\ \dot{\theta}_m & = & w \end{array} \right. \tag{6}$$

The control variable chosen for this system is the angular velocity w. The curvature's derivative with respect to the path's curvilinear abscissa, at the point T, is denoted as $g_c(s)$. Therefore,

$$\dot{c}_c(s(t)) = g_c(s(t))\dot{s}(t)$$

2.3 Model of a two-steering-wheels robot

Fig. 2 shows a geometric model of this mobile robot. The wheels' orientation angles are denoted as α and β. The distance between the two wheels is equal to l.

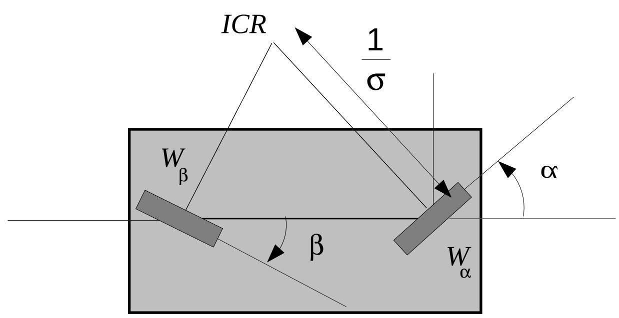


Figure 2: A two steering wheels robot

Denoting the velocities of points W_α and W_β as v_α and v_β respectively, the angular velocity of the vehicle's body with respect to the fixed frame is given by :

$$\dot{\theta}_m = \frac{1}{l}\left(v_\alpha sin\alpha - v_\beta sin\beta\right) \tag{7}$$

In the absence of skidding phenomena, v_α and v_β must satisfy the following constraint :

$$v_\alpha cos\alpha - v_\beta cos\beta = 0 \tag{8}$$

For this robot, the velocity coordinates of the point W_α are :

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = v \begin{pmatrix} cos\,(\theta_m + \alpha) \\ sin\,(\theta_m + \alpha) \end{pmatrix} \tag{9}$$

and, using the point W_α to characterize the vehicle's position, the following equations are obtained from (4), (7) and (9) :

$$\begin{cases} \dot{s} &= vcos\,(\theta_m - \theta_c + \alpha)\,/(1 - c_c y) \\ \dot{y} &= vsin\,(\theta_m - \theta_c + \alpha) \\ \dot{\theta}_m &= \frac{1}{l}\,(v_\alpha sin\alpha - v_\beta sin\beta) \end{cases} \tag{10}$$

under the constraint (8).

For path following purposes, the translational velocity of the vehicle's body can be prespecified. For example, it can be assumed that the velocity $v_\alpha(t)$ of the point W_α is predetermined. In this case, the control objective consists in regulating the lateral error y to zero and the angle $(\theta_m - \theta_c)$ between the vehicle's body and the tangent to the path at point T, to a desired angle denoted as θ_d. (For the sake of clarity of the control expressions, the variation of θ_d is supposed to depend on s, so that $\dot{\theta}_d(s(t)) = c_d(s(t))\dot{s}(t)$ and $\dot{c}_d(s(t)) = g_d(s(t))\dot{s}(t)$).

The control variables that may be used to achieve this objective are $\dot{\alpha}$, $\dot{\beta}$ and v_β. In fact, due to the constraint (8), the choice of v_β is usually not free and v_β can be deduced from $v_\alpha(t)$ according to the relation : $v_\beta = v_\alpha \frac{cos\alpha}{cos\beta}$. This leaves only two control variables, namely, $\dot{\alpha}$ and $\dot{\beta}$. However, if no precaution is taken, problems will clearly occur when using the above relation and when $cos\beta$ gets close to, or passes through zero. This difficulty closely meddles with a structural particularity of this type of vehicle which is that, depending on the relative orientation of the steering wheels, the vehicle may have either one or two degrees of mobility. More precisely, as long as the steering wheels are not parallel, the instantaneous motion of the vehicle's body is a pure rotation about the point I_{cr}, termed Instantaneous Center of Rotation, located at the intersection of the wheels' axles (*Descarte's Principle*, See [1]). When these wheels are parallel with $cos\alpha \neq 0$ and $cos\beta \neq 0$, the instantaneous motion of the vehicule's body is a pure translation in the direction of the wheels. This may still be seen as a pure rotation about a point located at infinity on one of the wheels' axles. Finally, in the singular configuration where the wheels are parallel with $cos\alpha = cos\beta = 0$, Descarte's Principle still applies, but any point on the line passing through the wheels'centers can potentially be an

Instantaneous Center of Rotation for the vehicle's body. This expresses the fact that, in this particular configuration, the mobile robot has two degrees of mobility, just like a unicycle-type vehicle.
The way chosen here to overcome this difficulty, consists, via a specific control strategy, in :

- *i)* imposing the additionnal constraint $cos\alpha = 0 \Leftrightarrow cos\beta = 0$, and
- *ii)* allowing only one degree of mobility when reaching the singular configuration $cos\alpha = cos\beta = 0$.

This strategy uses the fact that, away from the configuration $cos\alpha = cos\beta = 0$, the angular velocity of the vehicle's body may also be written as :

$$\dot{\theta}_m = v_\alpha \sigma \qquad \text{with} \qquad \sigma = \frac{1}{\overline{W_\alpha I_{cr}}} \tag{11}$$

When $cos\beta \neq 0$, σ can be deduced from α and β according to the relation :

$$\sigma = \frac{1}{l}(sin\alpha - tan\beta cos\alpha) \tag{12}$$

It is shown below that, through an adequate choice of $\dot{\beta}$ and v_β, and the introduction of an auxiliary variable, it is possible to extend the validity of the relation (11) to the case where $cos\beta = 0$, and fall upon a new control system for which the equality constraint (8) has been replaced by a simpler inequality constraint.

Choosing $\dot{\beta}$ and v_β

- Let us consider an auxiliary variable denoted as σ (the relation between this variable and the inverse of $\overline{W_\alpha I_{cr}}$ will be explicited further) and suppose that :

$$\mid \sigma \mid < \frac{1}{l} \tag{13}$$

  Then,

$$\sqrt{(l\sigma - sin\alpha)^2 + \cos^2 \alpha} \neq 0 \ \ \forall \alpha \tag{14}$$

Let β_d be the angle in $\left]\alpha - \frac{\pi}{2}, \alpha + \frac{\pi}{2}\right[$ such that :

$$cos\beta_d = \frac{cos\alpha}{\sqrt{(l\sigma - sin\alpha)^2 + \cos^2\alpha}} \quad \text{and} \quad sin\beta_d = \frac{sin\alpha - l\sigma}{\sqrt{(l\sigma - sin\alpha)^2 + \cos^2\alpha}} \tag{15}$$

this angle does exist in the interval $\left]\alpha - \frac{\pi}{2}, \alpha + \frac{\pi}{2}\right[$, since :

$$cos(\beta_d - \alpha) = cos\beta_d cos\alpha + sin\beta_d sin\alpha = \frac{1 - l\sigma sin\alpha}{\sqrt{(l\sigma - sin\alpha)^2 + \cos^2\alpha}}$$

is strictly positive, due to condition (13).

Using the following trivial identity :

$$\dot{\beta}_d = cos^2\beta_d \frac{d}{dt}\left[tan\beta_d\right]$$

and applying eqns (15), one gets :

$$\dot{\beta}_d = \frac{cos^2\alpha}{(l\sigma - sin\alpha)^2 + \cos^2\alpha} \frac{d}{dt}\left[\frac{sin\alpha - l\sigma}{cos\alpha}\right]$$

and so :

$$\dot{\beta}_d = \frac{\dot{\alpha}(1 - l\sigma sin\alpha) - l\dot{\sigma} cos\alpha}{(l\sigma - sin\alpha)^2 + \cos^2\alpha} \tag{16}$$

The control $\dot{\beta}$ is chosen so that β keeps tracking β_d. A simple solution is given by :

$$\dot{\beta} = \dot{\beta}_d - k_\beta\left(\beta - \beta_d\right)$$

with $k_\beta > 0$. According to this equation, if $\beta(0) = \beta_d(0)$, then $\beta(t) = \beta_d(t)$, $\forall t$.

In view of eqn (16), the complete expression of $\dot{\beta}$ can be written as :

$$\dot{\beta} = \frac{\dot{\alpha}(1 - l\sigma sin\alpha) - l\dot{\sigma} cos\alpha}{(l\sigma - sin\alpha)^2 + \cos^2\alpha} - k_\beta\left(\beta - \beta_d\right) \tag{17}$$

where :

- α and β are physical angles measured on the vehicle

- β_d is determined by eqns (15)
- σ and $\dot{\sigma}$ remain to be determined

This choice for $\dot{\beta}$ ensures that the error$(\beta - \beta_d)$ will remain small even if the measurements of α and β are slightly corrupted by noise.

- For v_β, we choose :

$$v_\beta = v_\alpha \sqrt{(l\sigma - sin\alpha)^2 + \cos^2\alpha} \tag{18}$$

so as to have, from eqns (15) :

$$v_\alpha cos\alpha - v_\beta cos\beta_d = 0$$

Since β tracks β_d closely, one is ensured, in this way, that the non-skidding constraint :

$$v_\alpha cos\alpha - v_\beta cos\beta = 0$$

is reasonably satisfied. In fact, it is theoretically exactly satisfied when $\beta(0) = \beta_d(0)$ and when the measurements of α and β are perfect.

The choice (18) for the control v_β is thus compatible with the constraint (8). Moreover, from eqns (15,18), we have :

$$v_\alpha sin\alpha - v_\beta sin\beta_d = v_\alpha l\sigma$$

and, using $\beta \equiv \beta_d$, one obtains :

$$v_\alpha sin\alpha - v_\beta sin\beta = v_\alpha l\sigma \tag{19}$$

In view of eqns (7) and (9), this shows that the auxiliary variable σ can physically be interpreted as the inverse of $\overline{W_\alpha I_{cr}}$. The knowledge of σ provides a way of determining the Instantaneous Center for Rotation in the singular configuration where $cos\alpha = cos\beta = 0$.

Now, by using eqn (19) in eqns (10), one obtains the following new control model :

$$\left\{ \begin{array}{lcl} \dot{s} & = & vcos\left(\theta_m - \theta_c + \alpha\right)/(1 - c_c y) \\ \dot{y} & = & vsin\left(\theta_m - \theta_c + \alpha\right) \\ \dot{\theta}_m & = & v\sigma \end{array} \right. \tag{20}$$

with $\dot{\alpha}$ and $\dot{\sigma}$, the new control variables. The only constraint on the choice of $\dot{\sigma}$ is that the inequality (13) must always be satisfied, σ being obtained by numerical integration of $\dot{\sigma}$.

Under the assumptions that α and β are exactly measured, and that the controls $\dot{\beta}$ and v_β, given by eqns (17) and (18), are exactly implemented, the model (20) is an accurate representation of the robot's kinematical behaviour. From there, it only remains to determine $\dot{\alpha}$ and $\dot{\sigma}$ in order to achieve the initial control objectives.

Remark 1
- *From eqn (18), v_β and v_α have the same sign.*
- *From eqns (15), it can be verified that σ, α, and β satisfy : $l\sigma cos\beta = sin(\alpha - \beta)$.*

3 Feedback linearization approach

The method has been introduced in [5] for a slightly different parametrization, and more recently in [8]; it consists in linearizing the vehicle's equations of motion in terms of some curvilinear abscissa parameter via an adequate feedback control law. In [5], this parameter represents the curvilinear abscissa drawn by the robot itself, while in [8], it represents the curvilinear abscissa, denoted here as s, along the tracked trajectory. The second option is here preferred because it causes less singularities in the control equations.
For the sake of legibility, the angular variables $\theta_m - \theta_c$ and $\theta_m - \theta_c - \theta_d$ will henceforth be replaced by θ and $\tilde{\theta}$ respectively.

3.1 Control of a unicycle-type robot

Following this approach, the equations of motion are expressed with respect to the new variable $\eta = \int_o^t \mid \dot{s} \mid d\tau$ instead of the time-index t. This variable has the physical meaning of the distance travelled by the vehicle along the path. Denoting $\frac{d}{d\eta}$ as $()'$, the system (6) may also be written :

$$\left\{ \begin{array}{lcl} s' & = & sign(v\frac{cos\theta}{1-c_c y}) \\ y' & = & tan\theta\,(1-c_c y)\,sign(v\frac{cos\theta}{1-c_c y}) \\ \theta' & = & \frac{w}{|v|}\mid \frac{1-c_c y}{cos\theta} \mid - c_c\, sign(v\frac{cos\theta}{1-c_c y}) \end{array} \right. \tag{21}$$

The control objective is to stabilize the output y at zero. Since the control w does not explicitly appear in the expression of y', a second derivation is needed. One then obtains after simple but tedious calculations :

$$y'' = \frac{w}{v cos^3\theta}(1-c_c y)^2 - c_c(1-c_c y)\frac{1+sin^2\theta}{cos^2\theta} - g_c y tan\theta \tag{22}$$

This equation is linearized by setting :

$$w = v\frac{cos\theta}{1-c_c y}\left[u\frac{cos^2\theta}{1-c_c y} + c_c(1+sin^2\theta) + g_c y\frac{cos\theta sin\theta}{1-c_c y}\right] \tag{23}$$

This gives :

$$y'' = u \tag{24}$$

The auxiliary control u must be calculated so as to fall upon a stable closed-loop system.

One may, for example, choose the following PD control law :

$$u = -k_{p_y} y - k_{v_y} y' \ \ ; k_{p_y} > 0 \ \ , \ \ k_{v_y} > 0 \tag{25}$$

From eqns (22) and (23), the resulting control is :

$$w = v\frac{cos\theta}{1-c_c y}\left\{y\frac{cos\theta}{1-c_c y}(g_c sin\theta - k_{p_y} cos\theta) + sin\theta\left[c_c sin\theta - k_{v_y} cos\theta sign\left(\frac{v cos\theta}{1-c_c y}\right)\right] + c_c\right\} \tag{26}$$

Using eqns (6,26), the closed-loop equations in the time domain are :

$$\left\{\begin{array}{lcl} \dot{s} & = & v\frac{cos\theta}{1-c_c y} \\ \dot{y} & = & v\, sin\theta \\ \dot{\theta} & = & v\frac{cos\theta}{1-c_c y}\left\{y\frac{cos\theta}{1-c_c y}(g_c sin\theta - k_{p_y} cos\theta) + sin\theta\left[c_c sin\theta - k_{v_y} cos\theta sign\left(\frac{v cos\theta}{1-c_c y}\right)\right]\right\} \end{array}\right. \tag{27}$$

Provided that some initial conditions are satisfied, it is shown below that solutions to the system (27) exist over $\Re^+$, and that y asymptotically converges to zero.

Let $c_{c_{max}}$ denote an upper bound of the path's curvature (i.e. $\mid c_c(s) \mid \leq c_{c_{max}} \ \forall s$), and assume that $\mid g_c(s) \mid$ is bounded. It can be stated that :

Proposition 1 *If the initial conditions $y(0)$ and $\theta(0)$ are such that $y^2(0) + \frac{tan^2(\theta(0))}{k_{p_y}} < \frac{1}{{c_{cmax}}^2}$ with $\mid \theta(0) \mid < \frac{\pi}{2}$, $v(t)$ and $\dot{v}(t)$ are bounded, and $v(t)$ does not converge to zero when t tends to infinity, then the solutions $y(t)$ and $\theta(t)$ of system (27) asymptotically converge to zero.*

Proof Proposition 1 is demonstrated by considering the Lyapunov function :

$$V = \frac{1}{2}\left(k_{p_y} y^2 + tan^2\theta(1 - c_c y)^2\right) \tag{28}$$

and assuming that the system's solutions exist over $[0, +\infty[$. Existence of the solutions may in turn be proved retrospectively after establishing that "explosion" phenomena cannot occur in finite time. This comes as a by-product of the present stability proof and the uniform bounds obtained for $y(t)$ and $\theta(t)$.
Calculating the time-derivative of this function along a system's solution gives :

$$\dot{V} = -k_{v_y}\left|v\frac{1 - c_c y}{cos\theta}\right| \sin^2\theta \leq 0 \tag{29}$$

with :

$$y^2 \leq \frac{2V}{k_{p_y}} \leq \frac{2V(0)}{k_{p_y}} < \frac{1}{c_{cmax}^2} - \epsilon \tag{30}$$

and :

$$(tan\theta)^2 \leq \frac{2V}{(1 - c_c y)^2} < K < +\infty \tag{31}$$

- For any system's solution, $V(t)$ being non-increasing converges to some limit value V_{lim}. By Barbalat's Lemma, $\dot{V}(t)$ being uniformly continuous converges to zero. As a consequence, and omitting the time index from now on, $vsin\theta$, and thus $v\theta$ (since $\mid \theta \mid < \theta_{sup} < \frac{\pi}{2}$), tend to zero.
- Since v is bounded, $v^2\theta$ also tends to zero. By differentiating $v^2\theta$, one finds that $\dot{\overline{v^2\theta}}$ is the sum of $-\frac{k_{p_y} v^3 y}{(1-c_c y)^2}$, which is uniformly continuous since its derivative is bounded, and of other terms which tend to zero.

- A slight generalization of Barbalat's lemma (see appendix) then tells us that $\overline{v^2\dot{\theta}}$ tends to zero. Therefore, v^3y, and thus vy, tend to zero.

- From the convergence of $v\theta$ and vy to zero, one deduces that vV, and thus vV_{lim}, tend to zero.

From there, it can be concluded that (y, θ) asymptotically converges to $(0, 0)$ if v does not.
(End of **proof**)

At this stage, two remarks can be made :

Remark 2 • *When $v(t)$ keeps the same sign, the convergence of y to zero is exponential with respect to η. This directly results from equations (24) and (25). If, in addition $|v(t)| > \epsilon > 0 \ \ \forall t$, then, $\lim_{t\to\infty} inf \left(\frac{\eta(t)}{t}\right) \geq \epsilon$ and the convergence is also exponential in time.*

- *If, initially, $\frac{\pi}{2} < \theta(0) < \frac{3\pi}{2}$, Proposition 1 (and its proof) still apply except that θ will now converge toward π. In the particular case where $\theta = \pm\frac{\pi}{2}$, then $\dot{\theta} = 0$ and y does not converge any longer to zero.*

3.2 Simulation results for a unicycle-type robot

Simulation of the control law (26) is presented for two reference trajectories : a straight line in Fig.3 and a semi-circle trajectory in Fig.4. In both cases, the initial values of $y(0)$ and $\theta(0)$ have been chosen so as to satisfy the conditions in Proposition 1.

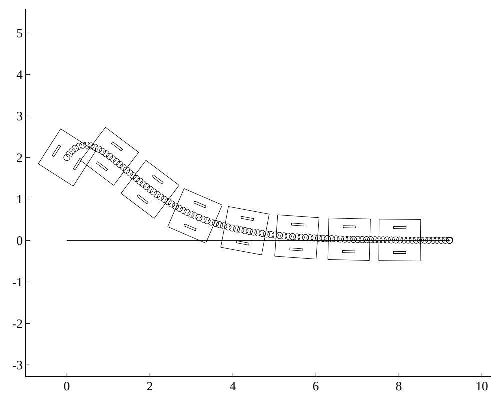


Figure 3: Straight line following

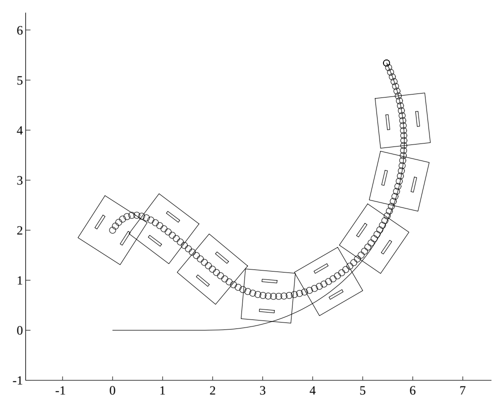


Figure 4: Circle following

3.3 Control of a two-steering-wheels robot

For the sake of legibility, $sin(\theta+\alpha)$, $cos(\theta+\alpha)$ are now denoted as $S_{\theta+\alpha}$ and $C_{\theta+\alpha}$ respectively.
In this case, it possible to take advantage of the two control variables to linearize the equations of two system outputs, chosen as y and $\tilde{\theta}$. Following the same approach as before, and in the same way as eqns (21) were derived from eqns (6), the system (20) yields the following equations :

$$\left\{\begin{array}{lcl} s' & = & sign(v\frac{C_{\theta+\alpha}}{1-c_c y}) \\ y' & = & \frac{S_{\theta+\alpha}}{C_{\theta+\alpha}}(1-c_c y)\, sign(v\frac{C_{\theta+\alpha}}{1-c_c y}) \\ \theta' & = & \sigma \mid \frac{1-c_c y}{C_{\theta+\alpha}} \mid sign(v) - c_c\, sign(v\frac{C_{\theta+\alpha}}{1-c_c y}) \end{array}\right. \tag{32}$$

Deriving y' a second time, one obtains :

$$y'' = \left(\sigma + \frac{\dot{\alpha}}{v}\right)\frac{(1-c_c y)^2}{C^3_{\theta+\alpha}} - c_c(1-c_c y)\frac{1+S^2_{\theta+\alpha}}{C^2_{\theta+\alpha}} - g_c y\frac{S_{\theta+\alpha}}{C_{\theta+\alpha}} \tag{33}$$

This equation is linearized by setting :

$$\dot{\alpha} = v\frac{C_{\theta+\alpha}}{1-c_c y}\left[u_y\frac{C^2_{\theta+\alpha}}{1-c_c y} + c_c(1+S^2_{\theta+\alpha}) + g_c y\frac{C_{\theta+\alpha}S_{\theta+\alpha}}{1-c_c y}\right] - v\sigma \tag{34}$$

This gives :

$$y'' = u_y \tag{35}$$

Choosing, as before, a PD control law for the auxiliary control variable u_y :

$$u_y = -k_{p_y}y - k_{v_y}y' \quad \text{with} \quad k_{p_y} > 0 \quad \text{and} \quad k_{v_y} > 0 \tag{36}$$

yields the control :

$$\begin{array}{rcl} \dot{\alpha} & = & v\frac{C_{\theta+\alpha}}{1-c_c y}\Big\{y\frac{C_{\theta+\alpha}}{1-c_c y}(g_c S_{\theta+\alpha} - k_{p_y}C_{\theta+\alpha}) + \\ & & S_{\theta+\alpha}\left[c_c S_{\theta+\alpha} - k_{v_y}C_{\theta+\alpha}sign\left(\frac{vC_{\theta+\alpha}}{1-c_c y}\right)\right] + c_c\Big\} - v\sigma \end{array} \tag{37}$$

Concerning the second output variable $\tilde{\theta} = \theta - \theta_d$ we have, from the last equation of (32) and the fact that $\dot{\theta}_d = c_d\dot{s}$:

$$\tilde{\theta}' = \left[\sigma\frac{(1-c_c y)}{C_{\theta+\alpha}} - (c_c + c_d)\right] sign\left(\frac{vC_{\theta+\alpha}}{1-c_c y}\right) \tag{38}$$

Differentiating this relation a second time in order to make $\dot{\sigma}$ explicitely appear, we get :

$$\begin{aligned}\tilde{\theta}'' &= \dot{\sigma}\frac{(1-c_cy)^2}{vC^2_{\theta+\alpha}} - (g_c+g_d) - \sigma\frac{1-c_cy}{C_{\theta+\alpha}}\left\{\frac{C_{\theta+\alpha}}{1-c_cy}y\left(g_cC_{\theta+\alpha}+k_{p_y}S_{\theta+\alpha}\right)+\right.\\ &\left.S_{\theta+\alpha}\left[c_cC_{\theta+\alpha}+k_{v_y}S_{\theta+\alpha}sign\left(\frac{vC_{\theta+\alpha}}{1-c_cy}\right)\right]\right\}\end{aligned} \tag{39}$$

By setting :

$$\begin{aligned}\dot{\sigma} &= v\frac{C_{\theta+\alpha}}{1-c_cy}\left\{\frac{C_{\theta+\alpha}}{1-c_cy}\left[u_\theta+(g_c+g_d)\right]+\sigma\left\{\frac{C_{\theta+\alpha}}{1-c_cy}y\left(g_cC_{\theta+\alpha}+k_{p_y}S_{\theta+\alpha}\right)+\right.\right.\\ &\left.\left.S_{\theta+\alpha}\left[c_cC_{\theta+\alpha}+k_{v_y}S_{\theta+\alpha}sign\left(\frac{vC_{\theta+\alpha}}{1-c_cy}\right)\right]\right\}\right\}\end{aligned} \tag{40}$$

we get the linear equation :

$$\tilde{\theta}'' = u_\theta$$

and choosing the auxiliary control u_θ as follows :

$$u_\theta = -k_{p_\theta}\tilde{\theta} - k_{v_\theta}\tilde{\theta}' \ , \ k_{p_\theta} > 0 \ , \ k_{v_\theta} > 0 \tag{41}$$

we obtain :

$$\begin{aligned}\dot{\sigma} &= v\frac{C_{\theta+\alpha}}{1-c_cy}\left\{\frac{C_{\theta+\alpha}}{1-c_cy}\left[-k_{p_\theta}\tilde{\theta}+(g_c+g_d)\right]+\sigma\left\{\frac{C_{\theta+\alpha}}{1-c_cy}y\left(g_cC_{\theta+\alpha}+k_{p_y}S_{\theta+\alpha}\right)+\right.\right.\\ &\left.\left.S_{\theta+\alpha}\left[c_cC_{\theta+\alpha}+k_{v_y}S_{\theta+\alpha}sign\left(\frac{vC_{\theta+\alpha}}{1-c_cy}\right)\right]\right\}-k_{v_\theta}\left[\sigma-(c_c+c_d)\frac{C_{\theta+\alpha}}{1-c_cy}\right]sign\left(\frac{vC_{\theta+\alpha}}{1-c_cy}\right)\right\}\end{aligned} \tag{42}$$

Regrouping eqns (20,37,42), the closed-loop system is characterized by the following equations :

$$\left\{\begin{aligned}\dot{s} &= v\frac{C_{\theta+\alpha}}{1-c_cy}\\ \dot{y} &= vS_{\theta+\alpha}\\ \dot{\theta} &= v(\sigma - c_c\frac{C_{\theta+\alpha}}{1-c_cy})\\ \dot{\alpha} &= v\frac{C_{\theta+\alpha}}{1-c_cy}[y\frac{C_{\theta+\alpha}}{1-c_cy}(g_cS_{\theta+\alpha}-k_{p_y}C_{\theta+\alpha})+\\ &\quad S_{\theta+\alpha}(c_cS_{\theta+\alpha}-k_{v_y}C_{\theta+\alpha}sign(\frac{1-c_cy}{C_{\theta+\alpha}}))+c_c]-v\sigma\\ \dot{\sigma} &= v\frac{C_{\theta+\alpha}}{1-c_cy}\left\{\frac{C_{\theta+\alpha}}{1-c_cy}\left[-k_{p_\theta}\tilde{\theta}+(g_c+g_d)\right]+\sigma\left\{\frac{C_{\theta+\alpha}}{1-c_cy}y\left(g_cC_{\theta+\alpha}+k_{p_y}S_{\theta+\alpha}\right)+\right.\right.\\ &\quad\left.\left.S_{\theta+\alpha}\left[c_cC_{\theta+\alpha}+k_{v_y}S_{\theta+\alpha}sign\left(\frac{vC_{\theta+\alpha}}{1-c_cy}\right)\right]\right\}-k_{v_\theta}\left[\sigma-(c_c+c_d)\frac{C_{\theta+\alpha}}{1-c_cy}\right]sign\left(\frac{vC_{\theta+\alpha}}{1-c_cy}\right)\right\}\end{aligned}\right. \tag{43}$$

We will show, as in section 3.1, that the controls (37) and (42) ensure the asymptotical convergence of y and $\tilde{\theta}$ to zero, provided that some initial conditions are satisfied.

To this purpose, let us consider the following Lyapunov functions :

$$V_y = \frac{1}{2}(k_{p_y} y^2 + y'^2) \tag{44}$$

and,

$$V_\theta = \frac{1}{2}(k_{p_\theta} \tilde{\theta}^2 + \tilde{\theta}'^{\,2}) \tag{45}$$

Proposition 2 *If the conditions :*

$$\sqrt{y(0)^2 + \frac{tan^2(\theta(0) + \alpha(0))}{k_{p_y}}} \leq r < \frac{1}{c_{c max}} \tag{46}$$

and :

$$\sqrt{2V_\theta(0)} + sup \mid c_c + c_d \mid < \frac{1}{l}(1 - c_{c max} r) \tag{47}$$

are satisfied, if v *and* $\dot{v}$ *are bounded, and if* v *does not converge to zero, then* $\mid \sigma(t) \mid$ *stays smaller than* $\frac{1}{l}$ *and the solutions* $y(t)$ *and* $\tilde{\theta}(t)$ *of system (43) asymptotically converge to zero.*

Proof

- The convergence of y and $\theta + \alpha$ to zero is proved exactly as in the case of proposition 1 by replacing θ by $\theta + \alpha$ and using the fact that V_y is non-increasing. Indeed, the time derivation of V_y (eqn (44)) gives :

$$\dot{V}_y = -k_{v_y} \left| v \frac{1 - c_c y}{C_{\theta+\alpha}} \right| S^2_{\theta+\alpha} \leq 0 \tag{48}$$

- The boundedness of $\mid \sigma(t) \mid$ is demonstrated as follows :

  - From the value of V_θ (eqn (45)), it can be deduced that :

$$\left[\sigma \frac{1 - c_c y}{C_{\theta+\alpha}} - (c_c + c_d) \right]^2 \leq 2V_\theta \tag{49}$$

  and thus :

$$\mid \sigma \mid\mid 1 - c_c y \mid \leq \sqrt{2V_\theta} + \mid c_c + c_d \mid \tag{50}$$

- Combining condition (46) and the fact that $\dot{V}_y \leq 0$ leads to $\mid y \mid < r$ and thus, $\mid 1 - c_c y \mid \geq 1 - c_{cmax} r > 0$. Introducing this result into inequality (50) yields $\mid \sigma \mid \leq \frac{\sqrt{2V_\theta} + |c_c + c_d|}{1 - c_{cmax} r}$.
- The time derivation of V_θ gives :

$$\dot{V}_\theta = -k_{v_\theta} \left[\sigma \frac{1 - c_c y}{C_{\theta+\alpha}} - (c_c + c_d) \right]^2 \left| v \frac{C_{\theta+\alpha}}{1 - c_c y} \right| \leq 0 \qquad (51)$$

thus, by applying condition (47), one gets that σ stays smaller than $\frac{1}{l}$.

- The convergence of $\tilde{\theta}$ to zero is proved through the following steps :
  - V_θ being positive and non-increasing asymptotically converges to some limit value. By Barbalat's Lemma, $\dot{V}_\theta$ converges to zero. As a consequence, $v\tilde{\theta}'$, $\dot{\tilde{\theta}}$, and thus $v[\sigma - (c_c + c_d)]$ tend to zero.
  - Since v is bounded, $v^2[\sigma - (c_c + c_d)]$ also tends to zero. By differentiating $v^2[\sigma - (c_c + c_d)]$, one finds that $\overline{(v^2[\sigma - (\dot{c}_c + c_d)])}$ is the sum of a term equivalent to $-v^3 k_{p_\theta} \tilde{\theta}$, which is uniformly continuous since its derivative is bounded, and other terms which tend to zero.
  - As in section 3.1, an extension of Barbalat's lemma then tells us that $\overline{(v^2[\sigma - (\dot{c}_c + c_d)])}$ tends to zero. Therefore, $v^3\tilde{\theta}$, and thus $v\tilde{\theta}$, tend to zero.
  - From the convergence of $v\tilde{\theta}$ and $v\tilde{\theta}'$ to zero, one deduces that vV_θ, and thus $vV_{\theta lim}$, tend to zero.
  - From there, it can be concluded that $\tilde{\theta}$ asymptotically converges to zero if v does not.

(End of **proof**)

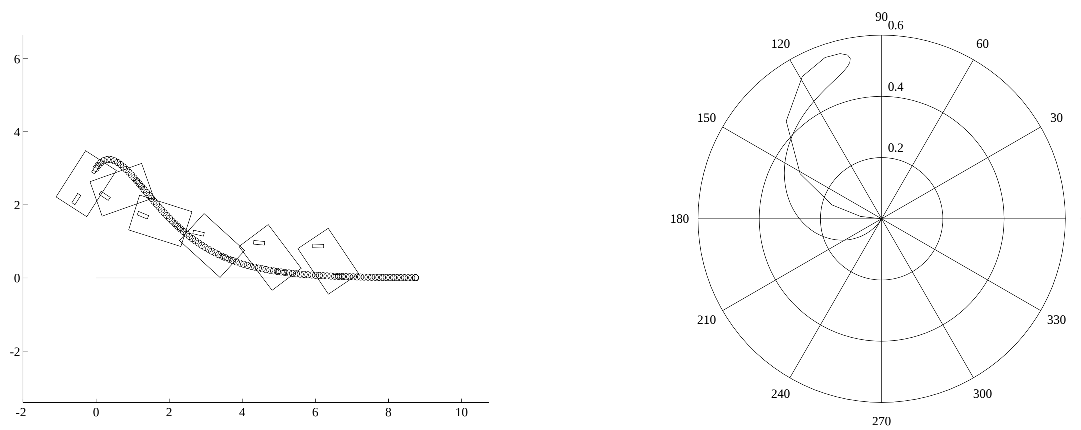
Remark 3 *As in the previous case, when $v(t)$ keeps the same sign, the convergence of y and $\tilde{\theta}$ are exponential with respect to η. If, in addition $\mid v(t) \mid > \epsilon > 0 \ \ \forall t$, then, the convergence is also exponential with respect to time.*

3.4 Simulation results for a two-steering-wheels robot

The system :

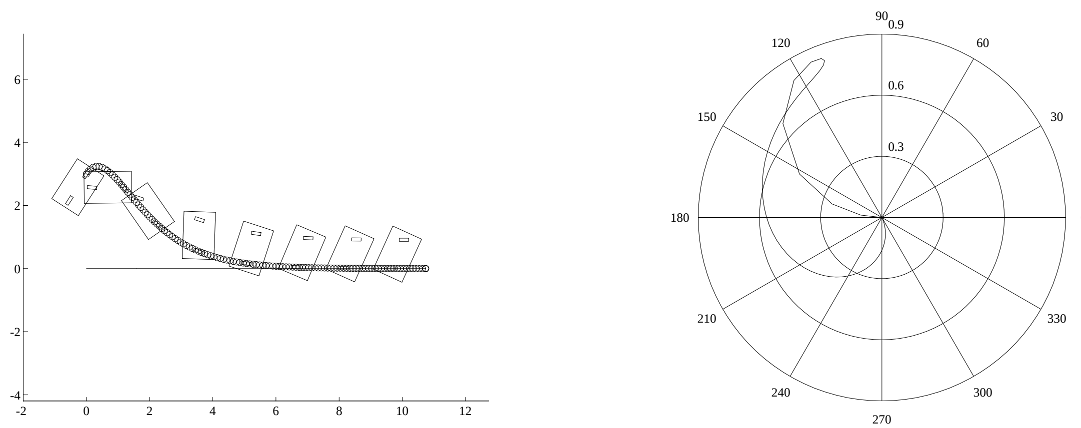
$$\left\{ \begin{array}{lcl} \dot{s} & = & v\frac{C_{\theta+\alpha}}{1-c_c y} \\ \dot{y} & = & vS_{\theta+\alpha} \\ \dot{\theta}_m & = & v\sigma \end{array} \right. \tag{52}$$

has been simulated with $\dot{\alpha}$ and $\dot{\sigma}$ taken as control variables. In Fig.5(a) and 6(a), the reference trajectory is a straight line and two different constant values are considered for θ_d ($\theta_d = -1\,rd$ and $\theta_d = -2\,rd$ respectively). In the third Fig.7(a), the reference trajectory is a circle, with θ_d set to $-1\,rd$.
For each simulation, the curve (σ, α) ((b) plot) is given in polar coordinates. It can be seen from these curves that $\mid \sigma \mid$ remains smaller than $\frac{1}{l}$ (with $l = 1$, here).



(a) Mobile trajectory (b) (σ, α) plot

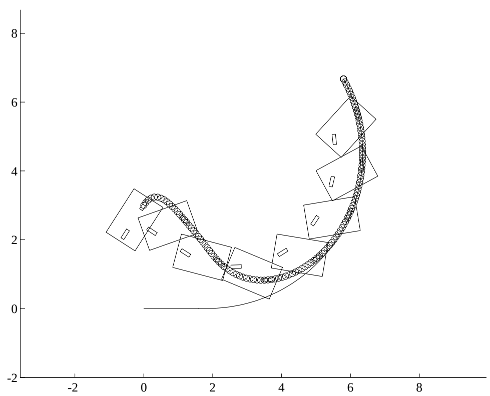
Figure 5: Following a straight line with $\theta_d = -1\,rd$



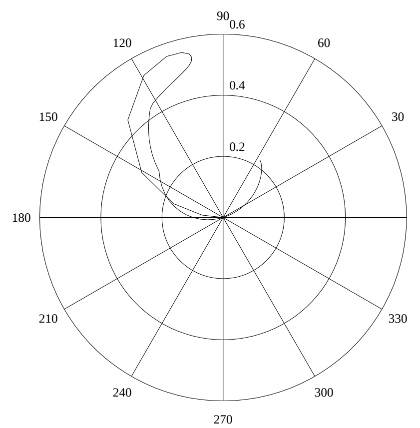
(a) Mobile trajectory

(b) (σ, α) plot

Figure 6: Following a straight line with $\theta_d = -2\,rd$



(a) Mobile trajectory



(b) (σ, α) plot

Figure 7: Following a circle with $\theta_d = -1\,rd$

4 Lyapunov-oriented control design

The feedback linearization technique used in the previous section presents the major advantage of yielding linear closed-loop output equations, the solutions of which are well known and can easily be tuned. Moreover, in the case of two-steering-wheels mobile robot, the transients of the two outputs of interest, namely y and $\tilde{\theta}$, are decoupled so that a perturbation action on one of them will not affect the other output.
On the other hand, the method has also some drawbacks. In particular, the domain of stability where the control behaves properly is not as large as it could be due to the existence of singularities. For instance, the orientation error $\mid \theta \mid$ in the case of the cart-type mobile robot, and the angle $\mid \theta + \alpha \mid$, in the case of the two-steering-wheels mobile robot, must not initially exceed $\frac{\pi}{2}$. This is shown in the convergence conditions given in propositions 1 and 2 which, although only sufficient and rather conservative, are somewhat limiting. It can be argued that the profile of the vehicles' trajectories, before convergence to the desired path and when initial position errors are large, resulting from a closed-loop linear equation in the form $y'' + k_{v_y} y' + k_{p_y} y = 0$, does not necessarily coincide with the user's idea of how the system should behave during transients. For example, it may be desirable to approximately maintain the angle θ, or $\theta + \alpha$, between the tangents to the vehicle's trajectory and the tracked trajectory, at a specified constant value when the vehicle is far from the desired trajectory. Such a requirement cannot easily be formulated within the framework of feedback linearization.
We will thus focus, in this section, on an alternative control design method which will overcome some of the shortcomings evoked above. It is based on the choice of adequate Lyapunov functions which are used not only for convergence analysis purposes but also at the control design stage.

4.1 Control of a unicycle-type robot

The method's principle has been proposed in [6] for unicycle-type mobile robots. The Lyapunov function considered in [6] was :

$$V = \frac{1}{2}\left[y^2 + \frac{1}{\lambda_\theta}\theta^2\right] \tag{53}$$

The method is here improved by considering a more general function :

$$V = \frac{1}{2}\left[f^2(y) + \frac{1}{\lambda_\theta}(\theta - \delta(y,v))^2\right] \tag{54}$$

where the functions $f(y) \ :\,]-r, r[\rightarrow \Re$ and $\delta(y,v) \ :\ \Re * \Re \rightarrow \Re$ are C^2 and C^1 respectively, and such that :

$H_1\ :\ f(\pm r) = \pm\infty$

$H_2\ :\ f(0) = \delta(0,v) = 0\ ,\forall v$

$H_3\ :\ {f_y}'(y) > 0\ ,\forall y$

$H_4\ :\ vf(y)sin(\delta(y,v)) \leq 0\ ,\forall y\ ,\forall v$

The functions f and δ are introduced in order to broaden the control stability domain and achieve additional user's objectives. For example, by requiring $f(y)$ to tend to infinity when $\mid y \mid$ tends to $\frac{1}{c_{cmax}}$, it is possible to keep $y(t)$ in the open interval $]-\frac{1}{c_{cmax}}, \frac{1}{c_{cmax}}[$ when $y(0)$ belongs to this interval. Concerning δ, it may be interpreted as the desired value for the orintation θ during transients.
Asymptotical convergence of y and θ to zero can be achieved by choosing a control w which makes V tend to zero. This is more precisely explained in the following proposition :

Proposition 3 *Consider the control :*

$$w = c_c\frac{vcos\theta}{1-c_c y} + {\delta_y}'vsin\theta + {\delta_v}'\dot{v} - \lambda_\theta f{f_y}'v\frac{sin\theta - sin\delta}{\theta - \delta} - k\lambda_\theta \mid v \mid (\theta - \delta)\ \ ;\ \ k > 0\ ,\ \lambda_\theta > 0 \tag{55}$$

which is applied to system (6).

If $\mid y(0) \mid < \frac{1}{c_{cmax}}$*, and if* $v(t)$ *and* $\dot{v}(t)$ *are bounded, and if* $v(t)$ *does not tend to zero when* t *tends to infinity, then the solutions* $y(t)$ *and* $\theta(t)$ *asymptotically converge to zero.*

Proof

- Differentiating V with respect to time gives :

$$\begin{aligned} \dot{V} &= ff_y{}'vsin\theta + \tfrac{1}{\lambda_\theta}(\theta-\delta)(\dot{\theta} - \delta_y{}'vsin\theta - \delta_v{}'\dot{v}) \\ &= ff_y{}'vsin\delta + (\theta-\delta)\left[ff_y{}'v\tfrac{sin\theta - sin\delta}{\theta-\delta} + \tfrac{1}{\lambda_\theta}\left(\dot{\theta} - \delta_y{}'vsin\theta - \delta_v{}'\dot{v}\right)\right] \end{aligned} \tag{56}$$

- Replacing $\dot{\theta}$ by $(w - c_c\frac{vcos\theta}{1-c_cy})$ with w given by eqn (55) yields :

$$\dot{V} = ff_y{}'vsin\delta - k \mid v \mid (\theta-\delta)^2 \leq 0 \tag{57}$$

  $V(t)$ is thus non-increasing, implying that $\mid f(y(t)) \mid$ and $\mid \theta(t) - \delta(y(t), v(t)) \mid$ are bounded. Thus, from the properties of f and δ, $\mid y(t) \mid < \frac{1}{c_{cmax}} - \epsilon$, $\forall t$, and $\mid \theta(t) \mid$ is bounded. $V(t)$ converges to some limit value, and, by Barbalat's Lemma, $\dot{V}(t)$ converges to zero since it is uniformly continuous. Therefore, in view of (57), $v(\theta-\delta)$ and $ff_y{}'vsin\delta$ (the time index is omitted to simplify the notations) asymptotically converge to zero.

- From the control expression (55) :

$$\dot{\overline{(\theta-\delta)}} = -\lambda_\theta ff_y{}'v\frac{sin\theta - sin\delta}{\theta-\delta} - k\lambda_\theta \mid v \mid (\theta-\delta) \tag{58}$$

  Hence :

$$\dot{\overline{v^2(\theta-\delta)}} = 2\dot{v}v(\theta-\delta) - \lambda_\theta ff_y{}'v^3\frac{sin\theta - sin\delta}{\theta-\delta} - k\lambda_\theta \mid v \mid v^2(\theta-\delta) \tag{59}$$

  The time-derivative of $v^2(\theta-\delta)$ is thus the sum of two terms which tend to zero and a third term which is uniformly continuous. Using the fact that $v^2(\theta-\delta)$ tends to zero, a slight extension of Barbalat's lemma then tells us that $\dot{\overline{v^2(\theta-\delta)}}$ also tends to zero. Therefore, $ff_y{}'v^3\frac{sin\theta - sin\delta}{\theta-\delta}$ tends to zero. This in turn implies, using the convergence of $v(\theta-\delta)$ to zero and the fact that $\left(\frac{sin\theta - sin\delta}{\theta-\delta}\right)^2 + (\theta-\delta)^2 > \epsilon > 0$ $(\forall\theta, \forall\delta)$, that $ff_y{}'v$ tends to zero.

  Recalling that $f_y{}'(y) > 0$ (from assumption H_3), and since $|y|$is bounded, we obtain that vf tends to zero.

- Now, from the convergence of $v(\theta-\delta)$ and vf to zero, v^2V tends likewise to zero with V converging to some limit value V_{lim}. Hence v^2V_{lim} tends to zero and, since v does not tend to zero, V_{lim} must be equal to zero.

  Therefore, $f(y)$ and $(\theta-\delta)$ tend to zero.

- From (H_2) and (H_3), y tends to zero. From (H_2) and the convergence of y to zero, δ also tends to zero.

  From the convergence of $(\theta-\delta)$ and δ to zero, θ tends to zero.

(End of **proof**)

Remark 4

- *According to the previous proposition, asymptotical stabilization of $y=0$ and $\theta=0$ can be obtained whenever $\mid y(0) \mid < \frac{1}{c_{cmax}}$. This initial condition is much weaker than the condition required when applying feedback linearization. In particular, $\mid \theta(0) \mid$ is no longer required to be smaller than $\frac{\pi}{2}$.*

- *$f(y)=y$ and $\delta=0$ are possible choices when $c_{cmax}=0$, i.e. when the tracked trajectory is a straight line. In this case, the convergence of y and θ to zero is achieved whatever the initial conditions.*

- *Differentiability of $\delta(y,v)$ with respect to v is not absolutely necessary. For instance, one may choose :*

$$\delta(y,v) = -sign(v)g_\delta(y) \tag{60}$$

with $g_\delta(y)$ being a C^1 function such that $g_\delta(0)=0$ and $yg_\delta(y)\geq 0\ \forall y$. A possible choice is, for example, the following sigmoid function :

$$g_\delta(y) = \theta_a \frac{e^{2k_\delta y}-1}{e^{2k_\delta y}+1} \quad ; k_\delta > 0 \ \ , \ 0 \leq \theta_a < \pi \tag{61}$$

In this case, $\mid \delta \mid$ is approximately equal to θ_a when y is away from zero.

- *According to relation (58), and in order to keep $(\theta-\delta)$ small, $f(y)$ should be chosen so as to maintain $f(y){f_y}'(y)$ as small as possible in the largest possible domain.*

For example :

$$f(y) = \frac{\frac{y}{k_1}}{\left(1 + \left(\frac{y}{k_2}\right)^2\right)^{\frac{1}{3}}} \quad ; k_1 > 0 \ , \ k_2 > 0 \tag{62}$$

($\Rightarrow f{f_y}' \rightarrow 0$ when $\mid y \mid \rightarrow +\infty$) when $c_{cmax} = 0$, and :

$$f(y) = \frac{\frac{g_y(y)}{k_1}}{\left(1 + \left(\frac{g_y(y)}{k_2}\right)^2\right)^{\frac{1}{3}}} \ , \ g_y(y) = \frac{r}{2} Log \frac{r+y}{r-y} \quad ; k_1 > 0 \ , \ k_2 > 0 \ , \ r > 0 \tag{63}$$

when $c_{cmax} \neq 0$.

- *When choosing f and δ according to eqns (60)-(63), linearization of the system's closed loop equations about $y = 0$ and $\theta = 0$ gives, when v keeps the same sign :*

$$y'' + k_{v_y} y' + k_{p_y} y = 0 \tag{64}$$

with :

$$\begin{array}{rcl} k_{v_y} & = & k_\delta \theta_a + k \lambda_\theta \\ k_{p_y} & = & \lambda_\theta \left(\frac{1}{{k_1}^2} + k k_\delta \theta_a\right) \end{array} \tag{65}$$

Equations (65) can be used to determine the constants λ_θ, k, k_δ and k_1 from prespecified values of k_{p_y} and k_{v_y}.

4.2 Simulation results for a unicycle-type robot

The application of control law (55) is illustrated through the following two figures (Fig.8 and Fig.9). Initial conditions were chosen in order to show that they can be now less constrained than they were within the linearization approach. The effect of the δ function is shown in Fig.8 where θ_a, the convergence angle, is set at $0.8rd$.

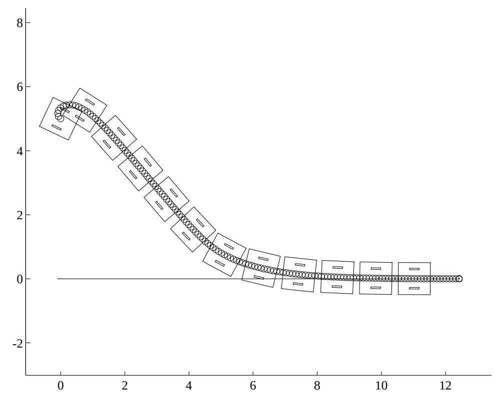


Figure 8: Straight line following

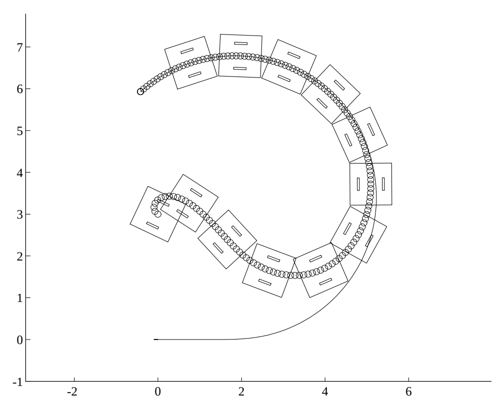


Figure 9: Circle following

4.3 Control of a two-steering-wheels robot

The control design is decomposed into two stages. The first one is related to the determination of the control $\dot{\alpha}$ for the regulation of the variables y and $\theta + \alpha$.
Applying the same method as in the case of the unicycle-type robot leads to considering the Lyapunov function :

$$V_y = \frac{1}{2}\left[f^2(y) + \frac{1}{\lambda_y}(\theta + \alpha - \delta)^2\right] \tag{66}$$

where f and δ satisfy the same conditions as before, and yields the following control law :

$$\dot{\alpha} = -v\left[\sigma - \delta_y{}' S_{\theta+\alpha} + c_c \frac{vC_{\theta+\alpha}}{1 - c_c y} + \lambda_y f_y' f \frac{S_{\theta+\alpha} - sin\delta}{\theta + \alpha - \delta}\right] - k_y \lambda_y |v|(\theta+\alpha-\delta) + \delta_v{}'\dot{v} \tag{67}$$

The second stage is related to the determination of the control $\dot{\sigma}$ for the regulation of $\tilde{\theta}$. A possible Lyapunov function to be considered at this stage is :

$$V_\theta = \frac{1}{2}\left[\tilde{\theta}^2 + \frac{1}{\lambda_\theta}\tilde{\sigma}^2\right] \tag{68}$$

where $\tilde{\sigma} = \sigma - \sigma_d$ and :

$$\sigma_d = \frac{c_c + c_d}{1 - c_c y} C_{\theta+\alpha} \tag{69}$$

The control $\dot{\sigma}$ may be chosen so as to have :

$$\dot{\tilde{\sigma}} = -v\lambda_\theta \tilde{\theta} - k_\theta(\sigma)\tilde{\sigma} \tag{70}$$

where $k_\theta(\sigma) \ :] - \frac{1}{l}, \frac{1}{l}[\to \Re$ is a strictly positive Lipschitz C^0 function such that :

- $k_\theta(\sigma) = k_\theta(-\sigma)$
- $k_\theta(\pm\frac{1}{l}) = +\infty$

This yields :

$$\begin{aligned}\dot{\sigma} \;=\; & v\tfrac{C_{\theta+\alpha}}{(1-c_c y)^2}\left\{(g_c+g_d)C_{\theta+\alpha}+(c_c+c_d)\left[g_c y\tfrac{C_{\theta+\alpha}}{1-c_c y}+c_c S_{\theta+\alpha}\right]\right\} \\ & -S_{\theta+\alpha}\tfrac{c_c+c_d}{1-c_c y}\left\{v\left[{\delta_y}' S_{\theta+\alpha}-\lambda_y f_y' f\tfrac{S_{\theta+\alpha}-sin\delta}{\theta+\alpha-\delta}\right]-k_y\lambda_y|v|(\theta+\alpha-\delta)+{\delta_v}'\dot{v}\right\} \\ & -v\lambda_\theta\tilde{\theta}-k_\theta\left(\sigma-\tfrac{c_c+c_d}{1-c_c y}C_{\theta+\alpha}\right)\end{aligned} \tag{71}$$

and :

$$\dot{V}_\theta = -\frac{k_\theta(\sigma)}{\lambda_\theta}\tilde{\sigma}^2 \leq 0 \tag{72}$$

Proposition 4 *Assume that $\theta_d(s)$ is chosen so that :*

$$\forall s \;:\; \mid c_c(s)+c_d(s)\mid < \frac{1}{l}(1-c_{cmax}r) \quad \textit{with} \quad r < \frac{1}{c_{cmax}} \tag{73}$$

and consider the controls (67) and (71) applied to the system (20). The functions $f(y)$, $\delta(y,v)$ and $k_\theta(\sigma)$ involved in the control expressions satisfy the set of properties defined above.

If $\mid y(0)\mid < r$ and $\mid \sigma(0)\mid < \frac{1}{l}$, then $\mid \sigma(t)\mid$ remains smaller than $\frac{1}{l}$ and, if $v(t)$ does not asymptotically converge to zero then the outputs $y(t)$, $(\theta+\alpha)(t)$ and $\tilde{\theta}(t)$ asymptotically converge to zero.

Proof

- Since V_y is non-increasing along any system's solution, $\mid f(y(t))\mid$ is bounded and $\mid y(t)\mid$ thus remains smaller than r.

  - Therefore, omitting the time-index to simplify the notations, there exists a positive real number ϵ such that :

    $$\mid 1-c_c y\mid > (1-c_{cmax}r)+\epsilon > 1-c_{cmax}r > 0$$

    and, from the condition (73) put on the choice of $\theta_d(s)$:

    $$|\sigma_d(t)| < \frac{1}{l}-\epsilon' \;,\; \forall t \quad \text{with} \quad \epsilon' = \frac{1}{l}\frac{\epsilon}{(1-c_{cmax}r)+\epsilon} \tag{74}$$

  - Asymptotical convergence of $y(t)$ and $(\theta+\alpha)(t)$ to zero is established exactly in the same way as $y(t)$ and $\theta(t)$ were proved to tend to zero in the case of the unicycle-type robot.

- Since V_θ is non increasing, $\mid \tilde{\theta}(t) \mid$ and $\mid \sigma(t) \mid$ are bounded along any system's solution
  - From eqn (70) :

$$\frac{1}{2}\frac{d}{dt}\tilde{\sigma}^2 = -v\lambda_\theta\tilde{\theta}\tilde{\sigma} - k_\theta(\sigma)\tilde{\sigma}^2 \tag{75}$$

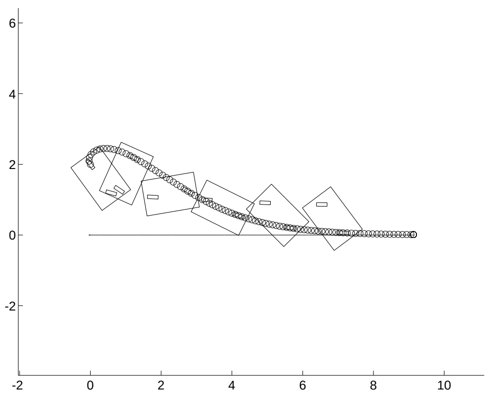
  - Since $k_\theta(\sigma)$ is continuous and grows to infinity when $\mid \sigma \mid$ tends to $\frac{1}{l}$, eqn (75) shows, in view of (74), that $\mid \sigma(t) \mid$ is bound to stay smaller than $\frac{1}{l}$. More precisely, there exists $\epsilon'' > 0$ such that $\mid \sigma(t) \mid < \frac{1}{l} - \epsilon''$, $\forall t$. This in turn implies that $k_\theta(\sigma(t))$ is bounded.
  - It remains to show that $\tilde{\theta}(t)$ tends to zero. Since V_θ is non increasing, V_θ converges to some limit value $V_{\theta lim}$ and, by Barbalat's Lemma, $\dot{V}_\theta$ tends to zero. Hence, from (72), $\tilde{\sigma}(t)$ tends to zero and $\tilde{\theta}(t)^2$ converges to $2V_{\theta lim}$. Also $\mid \tilde{\theta}(t) \mid^2 \leq 2V_\theta(0)$.
  - From eqn (70), since $\tilde{\sigma}$ tends to zero and $v\tilde{\theta}$ is uniformly continuous, $\dot{\tilde{\sigma}}$ and $v\tilde{\theta}$ tend to zero by application of the extended version of Barbalat's Lemma given in the Appendix.
  - The convergence of $\tilde{\sigma}$ and $v\tilde{\theta}$ to zero in turn implies that $v^2 V_{\theta lim}$ tends to zero and therefore, $V_{\theta lim} = 0$, since v does not (by assumption) converge to zero.
  - From there, it can be concluded that $\tilde{\theta}(t)$ converges to zero if v does not.

(End of **proof**).

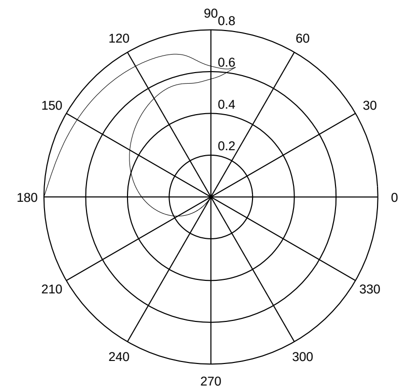
Remark 5 *By choosing a suitable function $k_\theta(\sigma, \alpha)$ that depends not only on σ but also on α, it is possible to ensure the satisfaction of the constraint $\sqrt{(l\sigma - sin\alpha)^2 + \cos^2\alpha} > \epsilon > 0$ without requiring the inequality $|\sigma| < \frac{1}{l}$ to be satisfied all the time.*

4.4 Simulation results for a two steering wheels robot

As in the case of the feedback linearization approach, three sets of curves are plotted. For each reference trajectory, a constant required orientation angle θ_d is specified, and for each simulation, the related curve (σ, α) in polar coordinates is drawn. In comparison to the feedback linearization approach, the contraints on the initial conditions are less severe.

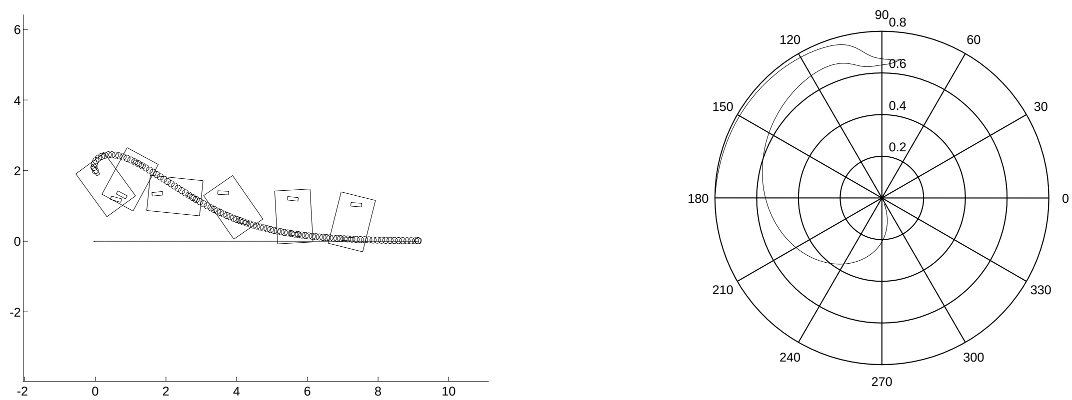


(a) Mobile trajectory



(b) (σ, α) plot

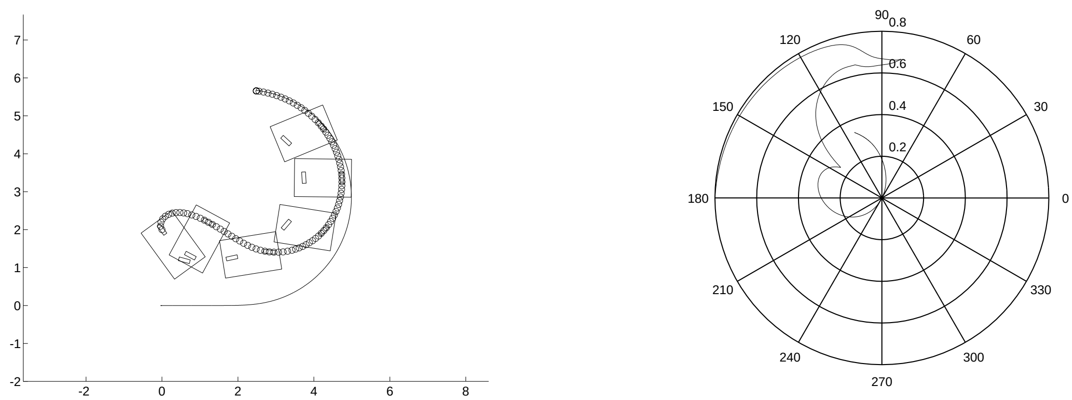
Figure 10: Following a straight line with $\theta_d = -1\,rd$



(a) Mobile trajectory

(b) (σ, α) plot

Figure 11: Following a straight line with $\theta_d = -2\,rd$



(a) Mobile trajectory

(b) (σ, α) plot

Figure 12: Following a circle with $\theta_d = -2\,rd$

5 Conclusion

In this paper, two types of controllers associated with a specific parametrization of the relative path-to-vehicle distance and orientation have been proposed for unicycle-type mobile robots and two-steering-wheels mobile robots. The first controller is designed to achieve exact output feedback linearization and decoupling, while the second one is obtained via a Lyapunov-oriented approach. For both controllers, path following convergence has been proved, provided that some conditions are satisfied. These conditions are less severe in the case of the second controller. The main difficulty arising in the case of a two-steering-wheels mobile robot is the passage through singular configurations where the wheels are parallel and their axles are colinear. A particular control strategy has been derived to overcome this difficulty. Extension of the results to mobile robots equipped with more than two steering wheels, although not explicited in the paper, is not difficult.

Appendix : Extension of Barbalat's Lemma

Lemma 1 *Let $f(t)$ and $g(t)$ be two function from $\Re^+$ to $\Re$ such that f is differentiable and g is uniformly continuous on $\Re^+$. If $\lim_{t\to\infty} f(t) = l$ and $\lim_{t\to\infty}(\dot{f}(t) - g(t)) = 0$, then $\lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} g(t) = 0$.*

Proof Assume that $\dot{f}(t)$ does not tend to zero, than $g(t)$ does not tend to zero either, meaning that :

$$\exists \epsilon > 0,\ \exists \{t_i\}_{i\in N}\ :\ \lim_{i\to+\infty} t_i = +\infty \quad \text{and} \quad \mid g(t_i) \mid > \epsilon,\ \ \forall i \tag{76}$$

Since $g(t)$ is uniformly continuous :

$$\exists \eta\ :\ 0 \leq \delta_i \leq \eta \Rightarrow \mid g(t_i + \delta_i) - g(t_i) \mid < \frac{\epsilon}{2} \tag{77}$$

and, from (76) and (77) :

$$\delta_i \in]0, \eta[\Rightarrow \mid g(t_i + \delta_i) \mid > \frac{\epsilon}{2} \tag{78}$$

From a classical averaging theorem :

$$\exists \delta_i \in]0, \eta[\ :\ \int_{t_i}^{t_i+\eta} g(t)\, dt = \eta\, g(t_i + \delta_i) \tag{79}$$

Thus, in view of (78) :

$$\left| \int_{t_i}^{t_i+\eta} g(t)\, dt \right| > \eta \frac{\epsilon}{2} \tag{80}$$

Introducing the function $h(t) = \dot{f}(t) - g(t)$ which, by assumption, tends to zero :

$$\exists t^*\ :\ t > t^* \Rightarrow \mid h(t) \mid < \frac{\epsilon}{4}$$

Hence :

$$t_i > t^* \Rightarrow \left| \int_{t_i}^{t_i+\eta} h(t)\, dt \right| < \eta \frac{\epsilon}{4} \tag{81}$$

On the other hand :

$$f(t_i + \eta) - f(t_i) = \int_{t_i}^{t_i+\eta} \dot{f}(t)\, dt = \int_{t_i}^{t_i+\eta} g(t)\, dt + \int_{t_i}^{t_i+\eta} h(t)\, dt$$

Therefore :

$$| f(t_i + \eta) - f(t_i) | = \left| \int_{t_i}^{t_i+\eta} g(t)\, dt \right| - \left| \int_{t_i}^{t_i+\eta} h(t)\, dt \right|$$

and, from (80) and (81) :

$$t_i > t^* \Rightarrow | f(t_i + \eta) - f(t_i) | > \eta \frac{\epsilon}{4}$$

This implies that $f(t)$ does not converge to some limit value, and this contradicts one of the Lemma's assumptions.
(End of **proof**)

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